# Generic hyperplane section of curves and an application to regularity bounds in positive characteristic

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**Abstract**

This paper investigates the Castelnuovo-Mumford regularity of the generic hyperplane section of projective curves in positive characteristic case, and yields an application to a sharp bound on the regularity for nondegenerate projective varieties.

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# 1 Introduction

The purpose of this paper is to study an upper bound of the index of regularity of a generic hyperplane section of projective curves and its application to sharp regularity bounds for projective varieties.

For a projective scheme $X \subset \mathbb{P}^N_K$, we define the Castelnuovo-Mumford regularity $\mathrm{reg}(X)$ as the smallest integer $m$ such that $\mathrm{H}^i(\mathbb{P}^N_K, \mathcal{I}_X(m-i)) = 0$ for all $i \geq 1$, see, e.g., [6]. The interest in this concept stems partly from the well-known fact: The regularity $\mathrm{reg}(X)$ is the smallest integer $m$ such that the minimal generators of the $n$-th syzygy module of the defining ideal $I$ of $X$ occur in degree $\leq m+n$ for all $n \geq 0$.

In particular, for a zero-dimensional scheme $S \subset \mathbb{P}^N_K$, we define the index of regularity $i(S)$ of $S$ as the smallest integer $t$ such that $\mathrm{H}^1(\mathbb{P}^N_K, \mathcal{I}_S(t)) = 0$. We remark that $\mathrm{reg}(S) = i(S) + 1$.

Throughout this paper, for a rational number $\ell \in \mathbb{Q}$, we write $\lceil \ell \rceil$ for the minimal integer which is larger than or equal to $\ell$, and $\lfloor \ell \rfloor$ for the maximal integer which is smaller than or equal to $\ell$.

Let $S \subset \mathbb{P}^N_K$ be a generic hyperplane section of a nondegenerate projective curve $C \subset \mathbb{P}^{N+1}_K$ over an algebraically closed field $K$. Then $S$ has the uniform position property in case $\mathrm{char}(K) = 0$, see [8], while the property does not necessarily hold in case $\mathrm{char}(K) > 0$, see [19]. Instead, even for the positive characteristic case, $S$ has the linear semi-uniform position property introduced in [1], see §2 for the definition. The linear semi-uniform position has an important role in studying the positive characteristic case.

For example, by studying the $h$-vectors of a zero-dimensional scheme $S$ in linear semi-uniform position, we have an upper bound on the index of regularity, that is, $i(S) \leq \lceil (\deg(S) - 1)/N \rceil$, see, e.g., [1, 18]. Also, there are some known facts on the sharpness of the above bound. If a zero-dimensional scheme $S \subset \mathbb{P}^N_K$ lies on a rational normal curve, then we have an equality, $i(S) = \lceil (\deg(S) - 1)/N \rceil$. On the other hand, we assume that a zero-dimensional scheme $S \subset \mathbb{P}^N_K$ is in uniform position and $\deg(S)$ is large enough. If the equality $i(S) = \lceil (\deg(S) - 1)/N \rceil$ holds, then $S$ lies on a rational normal curve, see, e.g., [14, 22].

In Section 2, we consider a generic hyperplane section $S \subset \mathbb{P}^N_K$ of a nondegenerate projective curve over an algebraically closed field $K$ such that $S$ does not have the uniform position property. So we always focus on the case $\mathrm{char}(K) > 0$. First, we will show that, under the condition that $N \geq 3$ and $\deg(S)$ is large enough, if $S$ does not have the uniform position property, then $i(S) \leq \lceil (\deg(S) - 1)/N \rceil - 1$ in (2.1) and (2.2). The lemmas are technically key results of this paper. As in classical Castelnuovo's method, we will show the assertion of the lemmas, and in fact, the linear semi-uniform position property will be useful for this proof. Then we apply the lemmas to the main result of this section, see Theorem 2.3. Let $S \subset \mathbb{P}^N_K$ be a generic hyperplane section of a nondegenerate projective curve with $\deg(S)$ large enough. Without assuming $S$ is in uniform position, if the equality $i(S) = \lceil (\deg(S) - 1)/N \rceil$ holds, then $S$ lies on a rational normal curve. Finally we describe a results on the index of regularity for a generic hyperplane section of very strange curves, see Proposition 2.6.

In Section 3, we study the Castelnuovo-Mumford regularity of projective varieties as an application of §2. In recent years upper bounds on the Castelnuovo-Mumford regularity of a projective variety $X \subset \mathbb{P}^N_K$ have been given by several authors in terms of $\dim(X)$, $\deg(X)$, $\mathrm{codim}(X)$ and $k(X)$, see, e.g., [10, 15, 18], where $k(X)$ is the Ellia-Migliore-Miró Roig number measuring the deficiency module, or sometimes called as the Rao module, see §3 for the definition. A regularity bound $\mathrm{reg}(X) \leq \lceil (\deg(X) - 1)/\mathrm{codim}(X) \rceil + \max\{k(X)\dim(X), 1\}$ is known for a nondegenerate projective variety $X$, see [15, 18]. Conversely, under the assumption that a nondegenerate projective variety $X$ is ACM, that is, the coordinate ring of $X$ is Cohen-Macaulay, if $\mathrm{reg}(X) \leq \lceil (\deg(X) - 1)/\mathrm{codim}(X) \rceil + 1$ and $\deg(X)$ is large enough, then $X$ is a variety of minimal degree, see [16, 20]. Moreover, there gives a classification of nondegenerate projective non-ACM varieties $X$ attaining a regularity bound $\mathrm{reg}(X) = \lceil (\deg(X) - 1)/\mathrm{codim}(X) \rceil + k(X)\dim(X)$. In [14], under the assumption that $\deg(X)$ is large enough and $\mathrm{char}(K) = 0$, it is shown that a projective non-ACM variety having the equality must be a curve on a rational ruled surface, that is, on a Hirzeburch surface. In §3, we show the corresponding result in the positive characteristic case as an application of (2.3), see Theorem 3.2.

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## 2 Regularity of a Generic Hyperplane Section of Projective Curves in Positive Characteristic

Let $K$ be an algebraically closed field with $\mathrm{char}(K) = p > 0$.

In this section we will show that if $S \subset \mathbb{P}^N_K$ is a generic hyperplane section of an integral curve with $\deg(S)$ large enough, then either $S$ is in uniform position or $i(S) \le \lceil (\deg(S) - 1)/N \rceil - 1$. Here the index of regularity $i(S)$ of $S$ is defined as the smallest integer $t$ such that $\mathrm{H}^1(\mathbb{P}^N_K, \mathcal{I}_S(t)) = 0$. (Notice that $\mathrm{reg}(S) = i(S) + 1 = a(R) + 2$, where $R$ is the coordinate ring of $S$ and $a(R)$ is an $a$-invariant of $R$, that is, $a(R) = \max\{\ell \mid [H^1_{\mathfrak{m}_R}(R)]_\ell \neq 0\}$ )

A zero-dimensional scheme $S \subset \mathbb{P}^N_K$ is called in uniform position if $\mathrm{H}_Z(t) = \max\{\deg(Z), \mathrm{H}_S(t)\}$ for all $t$, for any subscheme $Z$ of $S$, where $\mathrm{H}_Z$ and $\mathrm{H}_S$ denote the Hilbert function of $Z$ and $S$ respectively.

A zero-dimensional scheme $S$, spanning $\mathbb{P}^N_K$, is called in linear semi-uniform position if there are integers $v(i, S)$, simply written as $v(i)$, $0 \le i \le N$ such that every $i$-plane $L$ in $\mathbb{P}^N_K$ spanned by linearly independent $i+1$ points of $S$ contains exactly $v(i)$ points of $S$. A generic hyperplane section of a nondegenerate projective integral curve is in linear semi-uniform position, see [1]. We say $S$ is in linear general position if $v(i) = i + 1$ for all $i \ge 1$.

Let $S$ be a zero-dimensional scheme of $\mathbb{P}^N_K$ in linear semi-uniform position. Then $v(i+1) \ge (v(1) - 1)v(i) + 1$ for $0 \le i \le N - 1$, see [4]. Also, we have, by [1] (or see [18]), $i(S) \le \lceil (\deg(S) - 1)/N \rceil$.

Further, we note that "uniform position" implies "linear general position" and that "linear general position" implies "linear semi-uniform position".

**Lemma 2.1** *Let $S \subset \mathbb{P}^N_K$ be a generic hyperplane section of a nondegenerate projective integral curve $C \subset \mathbb{P}^{N+1}_K$ with $d = \deg(C)$. Assume that $N \geq 3$ $d \geq 25$. If $v(1) \geq 3$, then $i(S) \leq \lceil (d-1)/N \rceil - 1$.*

*Proof.* The assumption $v(1) \geq 3$ yields $v(i) \geq 2^{i+1} - 1$ for $0 \leq i \leq N$. Put $v = v(N-1)$ and $w = v(N-2)$. Note that $w \geq 2^{N-1} - 1$, $v \geq (v(1)-1)v(N-2) + 1 \geq 2w + 1$ and $d \geq 2v + 1 \geq 2^{N+1} - 1$.

We have only to show that $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^N_K}(\ell)) \to \mathrm{H}^0(\mathcal{O}_S(\ell))$ is surjective, where $\ell = \lceil (d-1)/N \rceil - 1$. For any fixed point $P \in S$, we will show that there is a union of $\ell$ hyperplanes $F = H(1) \cup \cdots \cup H(\ell)$ in $\mathbb{P}^N_K$ such that $S \cap F = S \backslash \{P\}$, as in the classical Castelnuovo's method for finite sets in linear general position.

First, let us take a hyperplane $H(1)$ which contains exactly $v$ points of $S \backslash \{P\}$ from the linear semi-uniform position property. Then $H(1)$ does not contain $P$.

Next, let us fix an $(N-2)$-plane $L$ in $H(1)$ such that $L$ contains exactly $w$ points of $S \cap H(1)$. Put $\ell_1 = \lfloor (d-v-1)/(v-w) \rfloor + 1$. Now we will inductively construct hyperplanes $H(2), \cdots, H(\ell_1)$ such that the number of points of $(S \backslash \{P\}) \cap (H(1) \cup \cdots \cup H(i))$ is $v + (i-1)(v-w)$ for $i = 1, \cdots, \ell_1$. In fact, since $d - 1 - v - (i-1)(v-w) \geq v - w$ for $i \leq \ell_1 - 1$, there exists a point $Q$ in $S \backslash (\{P\} \cup H(1) \cup \cdots \cup H(i))$ such that a hyperplane $M$ spanned by $L$ and $Q$ does not contain $P$. Then $M$ contains exactly $v - w$ points of $S \backslash (\{P\} \cup H(1) \cup \cdots \cup H(i))$ from the linear semi-uniform position property. So we take $H(i+1) = M$. Thus the union of $\ell_1$ hyperplanes $H(1) \cup \cdots \cup H(\ell_1)$ contains $v + (\ell_1 - 1)(v-w)$ points of $S$ and does not contain $P$. Also, we note that $S \backslash (\{P\} \cup H(1) \cup \cdots \cup H(\ell_1))$ consists of at most $v - w - 1$ points.

However, we see that $S \backslash (\{P\} \cup H(1) \cup \cdots \cup H(\ell_1))$ consists of exactly $v - w - 1$ points. In fact, if the number of the remaining points were less than $v - w - 1$, then the hyperplane spanned by $M$ and a point from $S \backslash (H(1) \cup \cdots \cup H(\ell_1))$ would contain at most $v - 1$ points of $S$, which contradicts with $v(N-1) = v$. Thus we also have that there exist a hyperplane $G$ containing the $(N-2)$-plane $L$, all the remaining points of $S \backslash (\{P\} \cup H(1) \cup \cdots \cup H(\ell_1))$ and the point $P$. Of course $S \cap G$ consists of exactly $v$ points including $P$.

Since $S \cap G$ is in linear semi-uniform position in $G \cong \mathbb{P}_K^{N-1}$, there are $\ell_2$ hyperplanes $M(\ell_1+1), \cdots, M(\ell_2)$ of $\mathbb{P}_K^{N-1}$ such that the union of them contains the remaining points and does not contain $P$, where $\ell_2 = \lceil (v-1)/(N-1) \rceil (= \lfloor (v-2)/(N-1) \rfloor + 1)$. Thus we can take $\ell_2$ hyperplanes $H(\ell_1+1), \cdots, H(\ell_2)$ of $\mathbb{P}_K^N$ as desired. Note that we used a fact from [1] that $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}_K^{N-1}}(t)) \to \mathrm{H}^0(\mathcal{O}_{S\cap G}(t))$ is surjective for all $t \geq \lceil (v-1)/(N-1) \rceil$, not necessarily for $t = \lceil (v-1)/(N-1) \rceil - 1$, without using the hypothesis of the induction on $N$. So, if necessary, we may need to take a (possibly reducible) hypersurface $F(1)$ of degree $\ell_2$ in place of the union of $\ell_2$ hyperplanes, and then go on the similar proof.

Therefore we have $S \cap (H(1) \cup \cdots \cup H(\ell_1) \cup \cdots \cup H(\ell_1+\ell_2)) = S \backslash \{P\}$ (or $S \cap (H(1) \cup \cdots \cup H(\ell_1) \cup F(1)) = S \backslash \{P\}$).

Thus the proof is reduced to an arithmetic question. In other words, we need to prove $\ell_1 + \ell_2 \leq \ell$, namely,

$$\left\lceil \frac{d-1}{N} \right\rceil - \left\lfloor \frac{d-v-1}{v-w} \right\rfloor - \left\lfloor \frac{v-2}{N-1} \right\rfloor \geq 3.$$

Moreover, from the above argument, we remark that $d = v + \ell_1(v-w)$.

First, assume that $N \geq 5$. Since $v - w \geq w + 1 \geq 4(N-1)$, it suffices to show that $(d-1)/N - (d-v-1)/4(N-1) - (v-2)/(N-1) \geq 3$. In fact, we easily have this inequality by reducing it to the case $d = 2v+1$. Hence we proved the case $N \geq 5$.

Second, assume that $N = 4$. The inequality $\lceil (d-1)/4 \rceil - \lfloor (d-v-1)/(v-w) \rfloor - \lfloor (v-2)/3 \rfloor \geq 3$ holds except for the case $(d, v, w) = (32, 15, 7)$ or $(33, 15, 7)$. But both cases contradict with $d = v + \ell_1(v-w)$. Hence we proved the case $N = 4$.

Finally, assume that $N = 3$. Then we have $\lceil (d-1)/3 \rceil - \lfloor (d-v-1)/(v-w) \rfloor - \lfloor (v-2)/2 \rfloor \geq 3$ except for the case $w = 3$ and $(d, v) = (25, 7), (25, 8), (25, 10), (25, 12), (28, 7)$ under the condition $d \geq 25$. But all the exceptional cases contradict with $d = v + \ell_1(v-w)$. Hence we proved the case $N = 3$.

$\square$

**Lemma 2.2** *Let $S \subset \mathbb{P}_K^N$ be a generic hyperplane section of a nondegenerate projective integral curve $C \subset \mathbb{P}_K^{N+1}$ with $d = \deg(C)$. Assume that $N \geq 3$ and $d \geq 23$. If $v(1) = 2$ and $v(2) \geq 4$, then $i(S) \leq \lceil (d-1)/N \rceil - 1$.*

*Proof.* In fact, by [2], the assumption in (2.2) yields that $\deg(C) = 2^k$ for some $k \geq N$ and $v(i, S) = 2^i$ for all $i \leq N-1$ since $d \geq 23$. In particular, $v(N-1) = 2^{N-1}$ and $v(N-2) = 2^{N-2}$.

First assume that $N \geq 5$. Just by copying the proof of (2.1) as in the Castelnuovo's method, we see that the proof is reduced to show an inequality $\lceil (2^k-1)/N \rceil - \lfloor (2^k - 2^{N-1} - 1)/(2^{N-1} - 2^{N-2}) \rfloor - \lfloor (2^{N-1} - 2)/(N-1) \rfloor \geq 3$, namely,

$$\left\lceil \frac{2^k - 1}{N} \right\rceil \geq 2^{k-N+2} - 1 + \left\lceil \frac{2^{N-1} - 1}{N-1} \right\rceil,$$

which is easily shown. Hence we proved the case $N \geq 5$.

Next assume that $N = 3$. As in the classical Castelnuovo's method, we will take a union of hyperplanes with containing $S$ and without containing $P$.

First let us take a hyperplane $H(1)$ with containing exactly 4 points of $S \backslash \{P\}$.

Now we will inductively construct hyperplanes $H(2), \cdots, H(\ell_1)$ such that the number of points of $(S\backslash\{P\}) \cap G(i)$ is $4i$ for $i = 1, \cdots, \ell_1$, where $\ell_1 = 2^{k-3}$ and $G(i) = H(1) \cup \cdots \cup H(i)$. For any $i = 1, \cdots, \ell_1 - 1$, we will show that there exists a hyperplane $H(i+1)$ with containing exactly 4 points of $S \backslash (\{P\} \cup G(i))$. In fact, take 2 points $Q_1$ and $Q_2$ in $S \backslash (\{P\} \cup G(i))$. Then there exists a point $Q_3$ from $S \backslash (\{P, Q_1, Q_2\} \cup G(i))$ such that the hyperplane spanned by $Q_1$, $Q_2$ and $Q_3$ does not contain any points of $S \cap (\{P\} \cup G(i))$, since the number of points of $S \backslash (\{P, Q_1, Q_2\} \cup G(i))$ is larger than that of $S \cap (\{P\} \cup G(i))$.

So the number of the remaining point of $S \backslash (\{P\} \cup H(1) \cup \cdots \cup H(\ell_1))$ is $2^{k-1} - 1$. Next we will inductively construct hyperplanes $H(\ell_1 + 1), \cdots, H(\ell_1 + \ell_2)$ for some $\ell_2 \leq \lceil (2^{k-1})/3 \rceil$, satisfying that $S \backslash \{P\} = S \cap (H(1) \cup \cdots \cup H(\ell_1 + \ell_2))$. In fact, assume that we already take hyperplanes $H(1), \cdots, H(i)$ for $i \geq \ell_1$ satisfying some suitable condition. If the number of the remaining points of $S \backslash (\{P\} \cup G(i))$, where $G(i) = H(1) \cup \cdots \cup H(i)$, is larger than 3, we can take the hyperplane $H(i+1)$ spanned by appropriate 3 points from

$S\backslash(\{P\}\cup G(i))$ such that $H(i+1)$ does not contain $P$. So the number of the points of $S\cap(H(i+1)\backslash G(i))$ is at least 3, and possibly 4. If the number of the remaining points of $S\backslash(\{P\}\cup G(i))$ is 3, then we take hyperplanes $H(i+1)$ and $H(i+2)$ such that $H(i+1)\cup H(i+2)$ contains the remaining 3 points of $S\backslash(\{P\}\cup G(i))$ and does not contain $P$. If the number of the remaining points of $S\backslash(\{P\}\cup G(i))$ is either 1 or 2, then we take a hyperplane $H(i+1)$ such that $H(i+1)$ contains the remaining 1 or 2 points of $S\backslash(\{P\}\cup G(i))$ and does not contain $P$.

Thus the proof is reduced to an arithmetic question as in (2.1). Namely, $\ell_1+\ell_2\leq\lceil(2^k-1)/3\rceil-1$, in other words,

$$\left\lceil\frac{2^k-1}{3}\right\rceil-2^{k-3}-\left\lceil\frac{2^{k-1}}{3}\right\rceil\geq 1.$$

Then we easily see the inequality except for the case $k=3,4$.

Hence we proved the case $N=3$.

Finally assume that $N=4$. Again we will prove as in the classical Castelnuovo's method.

First let us take hyperplane $H(1)$ with containing exactly 8 points of $S\backslash\{P\}$.

Now we will inductively construct hyperplanes $H(2),\cdots,H(\ell_1)$ for some integer $\ell_1\leq\lfloor(2^{k-1}+1)/7\rfloor$ such that $S\cap(H(i+1)\backslash G(i))$ contains at least 7 points and does not contain $P$, where $G(i)=H(1)\cup\cdots\cup H(i)$. In fact, take 2 points $Q_1$ and $Q_2$ from $S\backslash(\{P\}\cup G(i))$. Then there exists a point $Q_3$ in $S\backslash(\{P,Q_1,Q_2\}\cup G(i))$ such that the 2-plane $L$ spanned by $Q_1$, $Q_2$ and $Q_3$ does not contain any points of $S\cap(\{P\}\cup G(i))$ if the number of points of $S\backslash(\{P,Q_1,Q_2\}\cup G(i))$ is larger than that of $S\cap(\{P\}\cup G(i))$. In other words, we can take such $L$ if $S\backslash(\{P\}\cup G(i))$ contains at least $2^{k-1}+2$ points. Thus the 2-plane $L$ contains exactly 4 points of $S\backslash(\{P\}\cup G(i))$, and we put $S\cap L=\{Q_1,\cdots,Q_4\}$. Then there exists a point $Q_5$ from $S\backslash(\{P,Q_1,\cdots,Q_4\}\cup G(i))$ such that the hyperplane $M$ spanned by the point $Q_5$ and the 2-plane $L$ contains at least two points of $S\backslash(\{P,Q_1,\cdots,Q_4\}\cup G(i))$ without containing $P$, if the number of points of $S\backslash(\{P,Q_1,\cdots,Q_4\}\cup G(i))$ minus 2 is larger than that of $S\cap(\{P\}\cup G(i))$. In this case we put $H(i+1)=M$. In other words, we can go on this process if $S\backslash(\{P\}\cup G(i))$ contains at least $2^{k-1}+4$ points.

Thus we constructed a union of hyperplanes $G(\ell_1) = H(1) \cup \cdots \cup H(\ell_1)$ such that $G(\ell_1)$ contains at least $2^{k-1} - 4$ points of $S$ and does not contain $P$ for some $\ell_1 \leq \lfloor (2^{k-1} + 1)/7 \rfloor$.

So the number of the remaining point of $S \backslash (\{P\} \cup H(1) \cup \cdots \cup H(\ell_1))$ is at most $2^{k-1} + 3$. Next we will inductively construct hyperplanes $H(\ell_1 + 1), \cdots, H(\ell_1 + \ell_2)$ for some integer $\ell_2 \leq 2^{k-3} + 2$ satisfying that $S \backslash \{P\} = S \cap (H(1) \cup \cdots \cup H(\ell_1 + \ell_2))$. Assume that we already take hyperplanes $H(1), \cdots, H(i)$ for $i \geq \ell_1$ satisfying some suitable condition. If the number of the remaining points of $S \backslash (\{P\} \cup G(i))$, where $G(i) = H(1) \cup \cdots \cup H(i)$, is larger than 6, we can take a hyperplane $H(i+1)$ with containing at least 4 points of $S \backslash G(i)$ and without containing $P$. So the number of $S \cap (H(i+1) \backslash G(i))$ is at least 4, and possibly more. If the number of the remaining points of $S \backslash (\{P\} \cup G(i))$ is 6, then we take hyperplanes $H(i+1)$, $H(i+2)$ and $H(i+3)$ with $H(i+1) \cup H(i+2) \cup H(i+3)$ containing the remaining 6 points of $S \backslash (\{P\} \cup G(i))$ and without containing $P$. If the number of the remaining points of $S \backslash (\{P\} \cup G(i))$ is either 3, 4 or 5, then we take hyperplanes $H(i+1)$ and $H(i+2)$ with $H(i+1) \cup H(i+2)$ containing the remaining 3, 4 or 5 points of $S \backslash (\{P\} \cup G(i))$ and without containing $P$. If the number of the remaining points of $S \backslash (\{P\} \cup G(i))$ is either 1 or 2, then we take a hyperplane $H(i+1)$ with containing the remaining 1 or 2 points of $S \backslash (\{P\} \cup G(i))$ and without containing $P$. Thus we see that there exist hyperplanes $H(\ell_1 + 1), \cdots, H(\ell_1 + \ell_2)$ as desired.

Thus the proof is reduced to an arithmetic question as in (2.1). Namely, $\ell_1 + \ell_2 \leq \lceil (2^k - 1)/4 \rceil - 1$, in other words,

$$\left\lceil \frac{2^k - 1}{4} \right\rceil - \left\lfloor \frac{2^{k-1} + 1}{7} \right\rfloor - 2^{k-3} \geq 1.$$

Then we easily see the inequality.

Hence we proved the case $N = 4$. □

**Theorem 2.3** *Let $S \subset \mathbb{P}_K^N$ be a generic hyperplane section of a nondegenerate projective integral curve $C \subset \mathbb{P}_K^{N+1}$ with $d = \deg(C)$. If $d \geq \max\{N^2 + 2N + 2, 25\}$ and $i(S) = \lceil (d-1)/N \rceil$, then $S$ lies on a rational normal curve.*

*Proof.* For the case $N = 2$, the corresponding result as in [21, (3.2)] on the $h$-vector for the positive characteristic case is true, see [5, (1.1)] or [7, 9]. So the assertion follows from the proof of [14, (2.5)].

We may assume that $N \geq 3$ and that the Uniform Position Lemma fails for the curve $C$. Note that $d \geq 25$. Then, by [19, (2.5)], $C$ satisfies either (i) every secant of $C$ is a multisecant, that is, $v(1) \geq 3$, or (ii) every plane spanned by three points contains one more point of $C$, that is, $v(1) = 2$ and $v(2) \geq 4$. Therefore, by (2.1) and (2.2), we obtain that $i(S) \leq \lceil (d-1)/N \rceil - 1$. So we exclude the case.

Hence the assertion is proved. □

**Lemma 2.4** *Let $S \subset \mathbb{P}^2_K$ be a generic hyperplane section of a nondegenerate integral space curve $C$ with $d = \deg(C)$. If $v(1) \geq 4$, then $i(S) \leq \lceil (d-1)/2 \rceil - 1$.*

*Proof.* Put $v = v(1)$. Following the Castelnuovo's method, we will have the corresponding proof as in (2.1). For any fixed point $P \in S$, we have only to show that there is a union of $\ell$ lines $F = L(1) \cup \cdots \cup L(\ell)$ in $\mathbb{P}^N_K$ such that $S \cap F = S \backslash \{P\}$, where $\ell = \lceil (d-1)/2 \rceil - 1$.

First, let us take a line $L(1)$ which contains exactly $v$ points of $S \backslash \{P\}$ from the linear semi-uniform position property. Then $L(1)$ does not contain $P$.

Next, let us fix a point $Q$ of $L(1)$ and put $\ell_1 = \lfloor (d-v-1)/(v-1) \rfloor$. Then we can construct lines $L(2), \cdots, L(\ell_1)$, by taking inductively a line $L(i+1)$ with containing $Q$ and without containing any points of $(\{P\} \cup L(1) \cup \cdots \cup L(i)) \backslash \{Q\}$ for $1 \leq i \leq \ell_1 - 1$.

Moreover, since $S \backslash (\{P\} \cup L(1) \cup \cdots \cup L(\ell_1))$ consists of at most $v-2$ points (and in fact exactly $v-2$ points), we can take appropriate $v-2$ lines $L(\ell_1+1), \cdots, L(\ell_1+v-2)$ with containing the remaining points of $S \backslash \{P\}$ and without containing $P$.

Thus the proof is reduced to an arithmetic question. In other words, $\ell_1 + v - 2 \leq \ell$, namely, $\lceil (d-1)/2 \rceil - \lfloor (d-v-1)/(v-1) \rfloor - v + 1 \geq 0$, which is easily shown.

Hence the assertion is proved. □

**Lemma 2.5** *Let $S \subset \mathbb{P}^2_K$ be a generic hyperplane section of a nondegenerate integral space curve $C$ with $d = \deg(C)$. If $v(1) = 3$ and $d \geq 24$, then $i(S) \leq \lceil (d-1)/2 \rceil - 1$.*

*Proof.* Following the Castelnuovo's method, we will have the corresponding proof as in (2.2), the case $N = 3$. For any fixed point $P \in S$, we have only to show that there is a union of $\ell$ lines $F = L(1) \cup \cdots \cup L(\ell)$ in $\mathbb{P}^N_K$ such that $S \cap F = S \backslash \{P\}$, where $\ell = \lceil (d-1)/2 \rceil - 1$.

First, let us take a line $L(1)$ which contains exactly 3 points of $S \backslash \{P\}$ from the linear semi-uniform position property. Then $L(1)$ does not contain $P$.

Put $\ell_1 = \lfloor (d-4)/6 \rfloor + 1$. Then we can construct lines $L(2), \cdots, L(\ell_1)$, by taking inductively a line $L(i+1)$ without containing any points of $\{P\} \cup L(1) \cup \cdots \cup L(i)$ for $1 \leq i \leq \ell_1 - 1$.

Moreover, since $S \backslash (\{P\} \cup L(1) \cup \cdots \cup L(\ell_1))$ consists of at most $\lceil (d+1)/2 \rceil$ points, we can take appropriate $\ell_2$ lines $L(\ell_1 + 1), \cdots, L(\ell_2)$ with containing the remaining points of $S \backslash \{P\}$ and without containing $P$, where $\ell_2 = \lceil (d+3)/4 \rceil$.

Thus the proof is reduced to an arithmetic question. In other words, $\ell_1 + \ell_2 \leq \ell$, namely, $\lceil (d-1)/2 \rceil - \lfloor (d-4)/6 \rfloor - \lceil (d+3)/4 \rceil \geq 2$, which is easily shown for $d \geq 24$.

Hence the assertion is proved. □

We say that a nondegenerate projective integral curve $C$ is very strange if a generic hyperplane section $S$ of $C$ is not in linear general position.

**Proposition 2.6** *Let $S \subset \mathbb{P}^N_K$ be a generic hyperplane section of a nondegenerate projective integral curve $C \subset \mathbb{P}^{N+1}_K$. Assume that $C$ is very strange. If $d = \deg(C) \geq 25$, then $i(S) \leq \lceil (d-1)/N \rceil - 1$*

*Proof.* It immediately follows from (2.1), (2.2), (2.4), (2.5) and the proof of (2.3). □

# 3 An Application to a Sharp Bound on the Castelnuovo-Mumford Regularity

Let $K$ be an algebraically closed field. Let $S = K[x_0, \cdots, x_N]$ be the polynomial ring and $\mathfrak{m} = (x_0, \cdots, x_N)$ be the irrelevant ideal. Let $X$ be a projective scheme of $\mathbb{P}^N_K = \mathrm{Proj}(S)$. For an integer $m$, $X$ is said to be $m$-regular if $\mathrm{H}^i(\mathbb{P}^N_K, \mathcal{I}_X(m-i)) = 0$ for all $i \geq 1$. The Castelnuovo-Mumford regularity of $X \subset \mathbb{P}^N_K$ is the least such $m$ and is denoted by $\mathrm{reg}(X)$.

Let $k$ be a nonnegative integer. Then $X$ is called $k$-Buchsbaum if the graded $S$-module $\mathrm{M}^i(X) = \oplus_{\ell \in \mathbb{Z}} \mathrm{H}^i(\mathbb{P}^N_K, \mathcal{I}_X(\ell))$, called the deficiency module of $X$, is annihilated by $\mathfrak{m}^k$ for $1 \leq i \leq \dim(X)$, see, e.g., [12, 13]. On the other hand, $X$ is called strongly $k$-Buchsbaum if $X \cap V$ has the $k$-Buchsbaum property for any complete intersection $V$ of $\mathbb{P}^N_K$ with $\mathrm{codim}(X \cap V) = \mathrm{codim}(X) + \mathrm{codim}(V)$, possibly $V = \mathbb{P}^N_K$. So "strongly $k$-Buchsbaum" implies "$k$-Buchsbaum". Further we call the minimal nonnegative integer $n$, if there exists, such that $X$ is $n$-Buchsbaum (resp. strongly $n$-Buchsbaum), as the Ellia-Migliore-Miró Roig number (resp. the strongly Ellia-Migliore-Miró Roig number) of $X$ and denote by $k(X)$ (resp. $\bar{k}(X)$), see [14]. In case $X$ is not $k$-Buchsbaum for all $k \geq 0$, then we put $k(X) = \bar{k}(X) = \infty$. Note that $k(X) < \infty$ if and only if $\bar{k}(X) < \infty$. Moreover it is equivalent to saying that $X$ is locally Cohen-Macaulay and equi-dimensional.

Upper bounds on the Castelnuovo-Mumford regularity of a projective variety $X$ are given in terms of $\dim(X)$, $\deg(X)$, $\mathrm{codim}(X)$, $k(X)$ and $\bar{k}(X)$. Moreover, in case $\mathrm{char}(K) = 0$, the extremal cases for the bounds are classified under a certain assumption.

**Proposition 3.1** *Let $X$ be a nondegenerate projective variety in $\mathbb{P}^N_K$. Assume that $X$ is not ACM, that is, $k(X) \geq 1$. Then*

(*a*) $\mathrm{reg}(X) \leq \lceil (\deg(X) - 1)/\mathrm{codim}(X) \rceil + k(X) \dim(X)$.

(*b*) $\mathrm{reg}(X) \leq \lceil (\deg(X) - 1)/\mathrm{codim}(X) \rceil + \bar{k}(X) \dim(X) - \dim(X) + 1$.

*Furthermore, assume that* $\mathrm{char}(K) = 0$ *and* $\deg(X) \geq 2\,\mathrm{codim}(X)^2 + \mathrm{codim}(X) + 2$. *If the equality* $\mathrm{reg}(X) = \lceil (\deg(X) - 1)/\mathrm{codim}(X) \rceil + k(X) \dim(X)$ *holds, then $X$ is a curve on a rational ruled surface.*

*Proof.* See [14, 15, 18]. □

Now we will study the extremal case for the inequality in (3.1) in positive characteristic case. We assume that a variety is not ACM, see [16] for the ACM case.

**Theorem 3.2** *Let $X$ be a nondegenerate projective variety in $\mathbb{P}^N_K$ with $k(X) \geq 1$. Assume that either* $\mathrm{char}(K) = 0$ *and* $\deg(X) \geq \mathrm{codim}(X)^2 + 2\,\mathrm{codim}(X) + 2$*, or* $\mathrm{char}(K) = p > 0$ *and* $\deg(X) \geq \max\{2\,\mathrm{codim}(X)^2 + \mathrm{codim}(X) + 2, 25\}$.

(*a*) *If the equality* $\mathrm{reg}(X) = \lceil(\deg(X)-1)/\mathrm{codim}(X)\rceil + k(X)\dim(X)$ *holds, then $X$ is a curve on a rational ruled surface.*

(*b*) *If the equality* $\mathrm{reg}(X) = \lceil(\deg(X) - 1)/\mathrm{codim}(X)\rceil + \bar{k}(X)\dim(X) - \dim(X) + 1$ *holds, then $X$ is a curve on a rational ruled surface.*

*Proof.* We will prove (a). The proof of (b) is similar as in (a), which is left to the readers.

First we assume that $\mathrm{char}(K) = p > 0$ and $\deg(X) \geq \max\{2\,\mathrm{codim}(X)^2 + \mathrm{codim}(X) + 2, 25\}$. The lemmas (2.5), (2.6), (2.7) and (2.8) in [14] work for the case $\mathrm{char}(K) = p > 0$, although an assumption $\mathrm{char}(K) = 0$ is mentioned in [14]. However, for the positive characteristic case, we cannot apply [14, (2.5)] as an inductive step, because a generic hyperplane section of an integral curve is not necessarily in uniform position. In other words, the corresponding proof as in [14] works for the positive characteristic case, except for the Uniform Position Lemma.

Thus, by applying Theorem 2.3 in place of [14, (2.5)], we have the assertion.

On the other hand, for the case $\mathrm{char}(K) = 0$ and $\deg(X) \geq \mathrm{codim}(X)^2 + 2\,\mathrm{codim}(X) + 2$, we use [17, (3.3)] in place of [14, (2.6),(2.8)]. (Notice that [17, (3.3)] is a consequence of the "Socle Lemma", see [11], and cannot be applied for the positive characteristic case.) Hence we have the assertion. □

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